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APPROXIMATE SOLUTIONS OF NONLINEAR HEAT EQUATION FOR GIVEN FLOW*

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The one-dimensional problem of the nonlinear heat equation is considered. We assume that the heat flow in the origin of coordinates is the power function of time and the initial temperature is zero. Approximate solutions of the problem are given. Convergence of approximate solutions is discussed.

1 The problem statement. The nonlinear heat equation has the form [1]

$$\frac{\partial u}{\partial t} = \varkappa \frac{\partial}{\partial r}\left(u^n \frac{\partial u}{\partial r}\right), \qquad r > 0, \qquad t > 0 \tag{1}$$

where $u(r,t)$ is temperature in the point r; t is time; $\varkappa$ is coefficient of the heat conductivity.

Suppose the boundary condition has the form

$$u^n \frac{\partial u}{\partial r}\bigg|_{\theta=0} = -q_0 t^k, \qquad t > 0 \tag{2}$$

Where q_0 is constant. Equation (2) corresponds to the energy flow in the origin of coordinates.

We also take the initial condition in the form

$$u(r, t=0) = 0, \qquad r > 0 \tag{3}$$

The boundary value problem (1)—(3) was considered [1–4]. When the temperature specified on the boundary the problem for equation (1) were considered in [3–7]. Numerical solution of the problem (1)—(3) can be found using difference method [8].

In this work we are going to look for approximate solutions of the problem (1)—(3).

Using

$$u = v^{\frac{1}{n}}$$

we get

$$v_t = \varkappa v v_{rr} + \frac{\varkappa}{n} v_r^2, \qquad r > 0, \qquad t > 0 \tag{4}$$

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Taking the problem (1)—(3) into consideration we have the boundary and the initial conditions for equation (4) in the form

$$v^{\frac{1}{n}}\frac{\partial v}{\partial r}\bigg|_{\theta=0} = -q_0 n t^k, \qquad t > 0 \tag{5}$$

$$v(r, t=0) = 0, \qquad r > 0 \tag{6}$$

To solve the problem (4) — (6) we can use the self-similar variables [1–7, 9]

$$v(r,t) = At^m f(\theta), \qquad \theta = \frac{Br}{t^p} \tag{7}$$

where $f(\theta)$ is a function of θ; A, B, m and p are constants.
Assuming

$$p = \frac{m+1}{2}, \qquad \varkappa AB^2 = 1 \tag{8}$$

and substituting (7) into equation (4) we obtain

$$ff_{\theta\theta} + \frac{1}{n}f_\theta^2 + \frac{m+1}{2}\theta f_\theta - mf = 0 \tag{9}$$

Substituting (7) into the boundary condition (5) and taking into account

$$\frac{m(n+1)}{n} - \frac{(m+1)}{2} = k \tag{10}$$

$$q_0 B^{-1} n = A^{\frac{n+1}{n}} \tag{11}$$

we have the boundary condition for equation (9) in the form

$$f^{\frac{1}{n}}\frac{df}{d\theta}\bigg|_{\theta=0} = -1 \tag{12}$$

From equation (6) we find the second boundary condition

$$f(\theta \to \infty) = 0$$

Using conditions (8), (10) and (11) we get constants $A,\ B,\ m$ in form

$$A = \varkappa^{\frac{n}{n+2}} q_0^{\frac{2n}{n+2}} n^{\frac{2n}{n+2}}, \qquad B = \varkappa^{-\frac{n+1}{n+2}} q_0^{-\frac{n}{n+2}} n^{-\frac{n}{n+2}} \tag{13}$$

$$m = \frac{n(2k+1)}{n+2} \tag{14}$$

Taking expressions (8), (10) and (11) into account we have the boundary value problem (1)—(3) in the form

$$ff_{\theta\theta} + \frac{1}{n}f_\theta^2 + \frac{m+1}{2}\theta f_\theta - mf = 0 \tag{15}$$

$$f^{\frac{1}{n}}\frac{df}{d\theta}\bigg|_{\theta=0} = -1 \tag{16}$$

$$f(\theta \to \infty) = 0 \tag{17}$$

To find the solution of the problem (1) — (3) we have to solve the problem (15) — (17). This is the aim of this work.

2 Method applied. It is known that the velocity of the boundary for the nonlinear heat conductivity is finite. Let us assume that $\theta = \alpha$ is the boundary. Therefore, at the point α temperature is equal to zero ($f(\alpha) = 0$), but derivative is non-zero $\left(\dfrac{df}{d\theta} \neq 0 \right)$.

We look for approximate solution of the problem (15) — (17) in the form

$$f(\theta) = \sum_{i=0}^{N} \beta_i \left(\alpha - \theta\right)^i \tag{18}$$

From equation (18) we get

$$\left.\frac{df}{d\theta}\right|_{\theta=\alpha} = -\beta_1, \quad \left.\frac{d^2 f}{d\theta^2}\right|_{\theta=\alpha} = 2\beta_2, \quad \ldots \quad , \left.\frac{d^i f}{d\theta^i}\right|_{\theta=\alpha} = i!\left(-1\right)^i \beta_i. \tag{19}$$

Taking $f(\alpha) = 0$ into account we get $\beta_0 = 0$

Substituting (19) into equation (9) we have the coefficients β_i.

$$\begin{gathered}
\beta_1 = \frac{1}{2}\alpha n\left(m+1\right) \\
\beta_2 = \frac{1}{4}\frac{\left(m-1\right)n}{n+1} \\
\beta_3 = -\frac{1}{12}\frac{n(m-1)\left(nm+n+2m\right)}{\left(n+1\right)^2 \alpha\ \left(1+2\,n\right)\left(m+1\right)} \\
\beta_4 = \frac{1}{48}\frac{n(m-1)\left(nm+n+2m\right)P_4^{(m,n)}}{\alpha^2\left(n+1\right)^3\left(1+2\,n\right)\left(m+1\right)^2\left(3\,n+1\right)} \\
\beta_5 = -\frac{1}{240}\frac{n(m-1)\left(nm+n+2m\right)P_5^{(m,n)}}{\alpha^3\left(3\,n+1\right)\left(m+1\right)^3\left(1+2\,n\right)^2\left(n+1\right)^4\left(1+4\,n\right)}
\end{gathered} \tag{20}$$

where $P_4^{(m,n)}$ and $P_5^{(m,n)}$ are polynomials

$$P_4^{(m,n)} = 5\,nm - n + 7\,m - 3$$

$$\begin{gathered}
P_5^{(m,n)} = 303\,nm^2 + 82\,m^2 + 102\,n^3m^2 + 317\,n^2m^2 - 204\,n^2m \\
-238\,nm - 70\,m - 48\,n^3m + 12 + 7\,n^2 - 6\,n^3 + 31\,n
\end{gathered}$$

Using expression (20) we can obtain some exact solutions of the problem (15) — (17).

If $n + 2m + nm = 0$ then $\beta_i = 0 \quad (i \geq 3)$. Taking $m = -\frac{n}{n+2}$ into consideration we have exact solution

$$f(\theta) = \frac{1}{2}\alpha n\left(m+1\right)\left(\alpha-\theta\right) + \frac{1}{4}\frac{\left(m-1\right)n}{n+1}\left(\alpha-\theta\right)^2 \tag{21}$$

However this exact solution does not satisfy the boundary condition (16). This solution can be used to solve the Cauchy problem for equation (1).

If $m = 1$ then $\beta_i = 0 \quad (i \geq 2)$ and we have another exact solution for equation (15).

$$f(\theta) = \alpha n \, (\alpha - \theta) \tag{22}$$

Exact solution (22) satisfies the boundary condition (16).

3 Solutions of the problem (15) — (17). Consider the case $n > 0$ and $k = 1/n$, $(m = 1)$.

$$f(\theta) = \alpha n \, (\alpha - \theta) \tag{23}$$

Substituting (23) into the boundary condition (16) we obtain

$$\frac{(n+1)(\alpha^2 n)^{\frac{n+1}{n}}}{n\alpha} = 1 \tag{24}$$

From condition (24) we get the parameter α

$$\alpha = \left(n^{\frac{1}{n}}(n+1)\right)^{\frac{n}{n+2}} \tag{25}$$

The values of α and n at $k = 1/n$ are given in table 1.

Table 1:

n	1	4/3	2	5/2	3	4
α	1.2599	1.5299	2.0598	2.4586	2.8619	3.6840
n	9/2	5	11/2	6	13/2	7
α	4.1025	4.5256	4.9528	5.3838	5.8184	6.2561

Exact solutions of the boundary value problem (15) – (17) at $k = 1/n$ $(n > 0)$ are described by formula (23).

Consider the case $n = 1$ and $k > 0$. Approximate solution of the boundary value problem (15) – (17) can be written in the form

$$\begin{gathered} f(\theta) = \frac{1}{2}\alpha\,(m+1)\,(\alpha-\theta) + \frac{1}{8}(m-1)\,(\alpha-\theta)^2 \\ -\frac{1}{144}\frac{(3\,m+1)\,(m-1)}{(m+1)\,\alpha}(\alpha-\theta)^3 + \frac{1}{1152}\frac{(3\,m+1)\,(m-1)\,(3\,m-1)}{(m+1)^2\,\alpha^2}(\alpha-\theta)^4 \\ -\frac{1}{172800}\frac{(3\,m+1)\,(m-1)\,(201\,m^2-140\,m+11)}{(m+1)^3\,\alpha^3}(\alpha-\theta)^5 + \dots \end{gathered} \tag{26}$$

From the boundary condition (16) we obtain

$$\begin{gathered} \frac{\alpha^3}{2985984000}\frac{(105147\,m^4+384822\,m^3+519188\,m^2+307082\,m+66161)}{(m+1)^6} \\ \left(9491+24237\,m^4+84342\,m^3+54602\,m+103808\,m^2\right) = 1 \end{gathered} \tag{27}$$

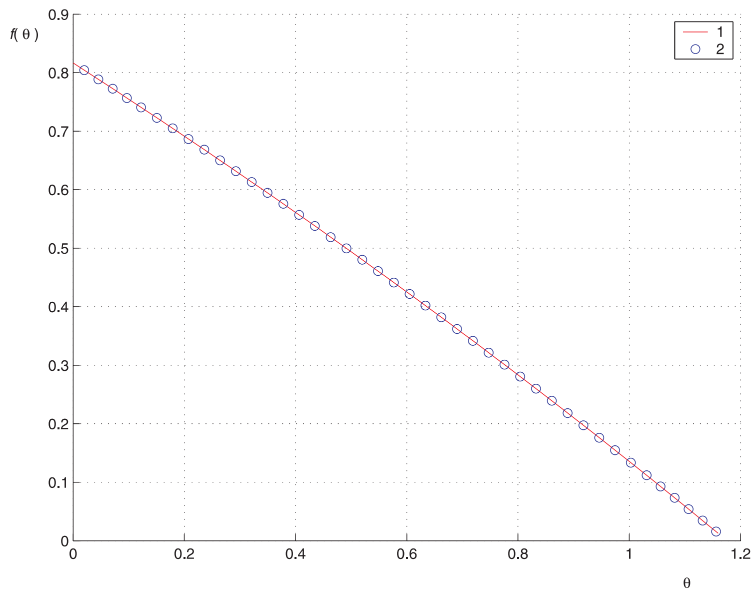

Figure 1: Comparison of approximate solution (28) and numerical solution of the problem (15) – (17): 1 — approximate solution; 2 — numerical solution.

From equation (27) we find

$$\alpha = 1440\,(m+1)^2 / \big(2548447839\,m^8 + 18195239088\,m^7 \\ +55955316456\,m^6 + 96920939400\,m^5 + 103409323126\,m^4 \\ +69458768096\,m^3 + 28562945760\,m^2 + 6527038184\,m + 627934051\big)^{\frac{1}{3}}$$

Table 2:

k	0	1	2	3	4	5
m	1/3	1	5/3	7/3	3	11/3
α	1.1762	0.7937	0.6222	0.5211	0.4532	0.4039

The values of α, m and k at $n = 1$ are given in table 2. Approximate solutions of the problem (15) – (17) at $n = 1$ are expressed by formula (26).

For the case $k = 0$ approximate solution of the boundary value problem (15) – (17) takes the form

$$f(\theta) = \begin{cases} \dfrac{2\alpha(\alpha-\theta)}{3} - \dfrac{(\alpha-\theta)^2}{12} + \dfrac{(\alpha-\theta)^3}{144\alpha} - \dfrac{(\alpha-\theta)^5}{23040\alpha^3}, & \\ \hfill 0 < \theta < \alpha; & \\ 0, \hfill \alpha < \theta; & \end{cases} \tag{28}$$

where α is $\alpha = 1.1762$.

To check approximate solution (28) we have compared it with numerical solution of the boundary value problem (15) – (17). The comparison of approximate solution (28) and numerical one at $k = 0$ and $n = 1$ is given on Fig.1. Solid line

is approximate solution and circles correspond to numerical solution of the problem (15) – (17). From Fig.1 we can see that these solutions are similar.

Consider the case $n = 4/3$ and $k > 0$. Approximate solution of the boundary value problem (15) – (17) is given by formula

$$
\begin{aligned}
f(\theta) = {} & \frac{2}{3}\alpha\,(m+1)\,(\alpha-\theta) + \frac{1}{7}\,(m-1)\,(\alpha-\theta)^2 \\
& - \frac{2}{539}\,\frac{(5\,m^2 - 2 - 3\,m)\,(\alpha-\theta)^3}{\alpha\,(m+1)} \\
& + \frac{1}{37730}\,\frac{(205\,m^3 - 188\,m^2 - 43\,m + 26)\,(\alpha-\theta)^4}{(m+1)^2\,\alpha^2} \\
& - \frac{9\,(29055\,m^4 + 7078\,m + 1199\,m^2 - 464 - 36868\,m^3)\,(\alpha-\theta)^5}{137997475\alpha^3\,(m+1)^3} + \ldots
\end{aligned} \tag{29}
$$

Some values of α, m and k at $n = 4/3$ are given in table 3.

Table 3:

k	0	1	2	3	4	5
m	2/5	6/5	2	14/5	18/5	22/5
α	0.9256	0.6000	0.4608	0.3807	0.3278	0.2897

Consider the case $k = 0$ and $n > 0$. From the expression (14) we have

$$m = \frac{n}{n+2}$$

Approximate solutions of the boundary value problem (15) – (17) at $k = 0$, $n > 0$ take the form

$$
\begin{aligned}
f(\theta) = {} & \frac{1}{2}\left(\frac{\alpha\,n^2}{n+2} + \alpha\,n\right)(\alpha-\theta) - \frac{1}{2}\,\frac{n\,(\alpha-\theta)^2}{(n+1)\,(n+2)} + \\
& + \frac{1}{6}\,\frac{n^2\,(\alpha-\theta)^3}{\alpha\,(n+1)^3\,(1+2\,n)} - \frac{1}{24}\,\frac{(2\,n^2 + n - 3)\,n^2\,(\alpha-\theta)^4}{(n+1)^5\,(1+2\,n)\,\alpha^2\,(3\,n+1)} + \\
& + \frac{1}{120}\,\frac{n^2\,(12 + 8\,n - 75\,n^2 + 12\,n^5 - 77\,n^3)\,(\alpha-\theta)^5}{\alpha^3\,(n+1)^7\,(1+2\,n)^2\,(3\,n+1)\,(1+4\,n)} + \ldots
\end{aligned} \tag{30}
$$

Using boundary condition (16) we obtain the value of α. Some values of α and n at $k = 0$ are given in table 4

Table 4:

n	1	4/3	2	5/2	3	4
α	1.4819	1.1578	0.7889	0.6283	0.5178	0.3775

n	9/2	5	11/2	6	13/2	7
α	0.3307	0.2934	0.2632	0.2382	0.2172	0.1994

4 Conclusion. The boundary value problem of the nonlinear heat equation for the given flow was considered. This problem was solved using the both numerical and analytical approaches. Some exact solutions were found. Approximate solutions of the boundary value problem were obtained. Comparison of the numerical and the approximate solutions was given.

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